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The Optimal Stopping Problem for Fuzzy Random Sequences

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1. Introduction and notations

Fuzzy random variables was first studied by Puri and Ralescu [7] and have been studied by many authors. Stojaković [9] discussed fuzzy conditional expectation and Puri and Ralescu [8] studied fuzzy martingales. This paper discusses optimal stopping problems of a sequence of fuzzy random variables.

Let $(\Omega, \mathcal{M}, P)$ be a probability space, $\mathcal{M}$ is a σ-field and P is a probability measure. Let $\mathbf{R}$ be the set of all real numbers and let $\mathbf{N}$ be the set of all nonnegative integers. $\mathcal{B}$ denotes the Borel σ-field of $\mathbf{R}$ and $\mathcal{I}$ denotes the set of all bounded closed sub-intervals of $\mathbf{R}$. A fuzzy set $\tilde{a}$ is called a fuzzy number if the membership function $\tilde{a}: \mathbf{R} \mapsto [0,1]$ is normal, upper-semicontinuous, convex and has a compact support. $\mathcal{R}$ denotes the set of all fuzzy numbers. We write the α-cut ($\alpha \in [0,1]$) of a fuzzy number $\tilde{a} \in \mathcal{R}$ by

$$\tilde{a}_\alpha := [\tilde{a}_\alpha^-, \tilde{a}_\alpha^+], \quad \alpha \in [0,1].$$

A map $\tilde{X}: \Omega \mapsto \mathcal{R}$ is called a fuzzy random variable if

$$\{(\omega, x) \mid \tilde{X}(\omega)(x) \geq \alpha\} = \{(\omega, x) \mid x \in \tilde{X}_\alpha(\omega)\} \in \mathcal{M} \times \mathcal{B} \quad \text{for all } \alpha \in [0,1], \tag{1.2}$$

where $\tilde{X}_\alpha(\omega) = [\tilde{X}_\alpha^-(\omega), \tilde{X}_\alpha^+(\omega)] := \{x \in \mathbf{R} \mid \tilde{X}(\omega)(x) \geq \alpha\} (\in \mathcal{I})$ is α-cut of fuzzy numbers $\tilde{X}(\omega)$ for $\omega \in \Omega$.

Lemma 1.1 ([10, Theorems 2.1 and 2.2]). For a map $\tilde{X}: \Omega \mapsto \mathcal{R}$, the following (i) and (ii) are equivalent:

(i) $\tilde{X}$ is a fuzzy random variable.

(ii) The maps $\omega \mapsto \tilde{X}_\alpha^-(\omega)$ and $\omega \mapsto \tilde{X}_\alpha^+(\omega)$ are measurable for all $\alpha \in [0,1]$.

A fuzzy random variable $\tilde{X}$ is called integrably bounded if $\omega \mapsto \tilde{X}_\alpha^-(\omega)$ and $\omega \mapsto \tilde{X}_\alpha^+(\omega)$ are integrable for all $\alpha \in [0,1]$. For an integrably bounded fuzzy random variable $\tilde{X}$, we define closed intervals

$$E(\tilde{X})_\alpha = \left[\int_\Omega \tilde{X}_\alpha^-(\omega)\, \mathrm{d}P(\omega), \int_\Omega \tilde{X}_\alpha^+(\omega)\, \mathrm{d}P(\omega)\right], \quad \alpha \in [0,1]. \tag{1.3}$$

Then the map $\alpha \mapsto E(\tilde{X})_\alpha$ is left-continuous by the dominated convergence theorem. Therefore, the expectation $E(\tilde{X})$ is a fuzzy number defined by

$$E(\tilde{X})(x) := \sup_{\alpha \in [0,1]} \min\left\{\alpha, 1_{E(\tilde{X})_\alpha}(x)\right\} \quad \text{for } x \in \mathbf{R}. \tag{1.4}$$

For an integrably bounded fuzzy random variable $\tilde{X}$ and a sub-σ-field $\mathcal{N}(\subset \mathcal{M})$, the conditional expectation $E(\tilde{X}|\mathcal{N})$ is defined as follows : For $\alpha \in [0,1]$, there exist unique classical conditional expectations $E(\tilde{X}_\alpha^-|\mathcal{N})$ and $E(\tilde{X}_\alpha^+|\mathcal{N})$ such that

$$\int_\Lambda E(\tilde{X}_\alpha^-|\mathcal{N})(\omega)\,\mathrm{d}P(\omega) = \int_\Lambda \tilde{X}_\alpha^-(\omega)\,\mathrm{d}P(\omega) \quad \text{for all } \Lambda \in \mathcal{N}, \tag{1.5}$$

and

$$\int_\Lambda E(\tilde{X}_\alpha^+|\mathcal{N})(\omega)\,\mathrm{d}P(\omega) = \int_\Lambda \tilde{X}_\alpha^+(\omega)\,\mathrm{d}P(\omega) \quad \text{for all } \Lambda \in \mathcal{N}. \tag{1.6}$$

Then we can easily check the maps $\alpha \mapsto E(\tilde{X}_\alpha^-|\mathcal{N})(\omega)$ and $\alpha \mapsto E(\tilde{X}_\alpha^+|\mathcal{N})(\omega)$ are left-continuous by the monotone convergence theorem. Therefore, we define

$$E(\tilde{X}_\alpha|\mathcal{N})(\omega) := [E(\tilde{X}_\alpha^-|\mathcal{N})(\omega), E(\tilde{X}_\alpha^+|\mathcal{N})(\omega)] \quad \text{for } \omega \in \Omega. \tag{1.7}$$

and we give a conditional expectation by a fuzzy random variable

$$E(\tilde{X}|\mathcal{N})(\omega)(x) := \sup_{\alpha\in[0,1]} \min\left\{\alpha, 1_{E(\tilde{X}_\alpha|\mathcal{N})(\omega)}(x)\right\} \quad \text{for } x \in \mathbf{R}. \tag{1.8}$$

2. An optimal stopping problem

Let $\{\tilde{X}_n\}_{n\in\mathbf{N}}$ be a sequence of fuzzy random variables. $\mathcal{M}_n$ $(n \in \mathbf{N})$ denotes the smallest σ-field on Ω generated by $\{\tilde{X}_{k,\alpha}^-, \tilde{X}_{k,\alpha}^+ \mid k = 0,1,2,\cdots,n; \alpha \in [0,1]\ \}$, and $\mathcal{M}_\infty$ denotes the smallest σ-field generated by $\bigcup_{n\in\mathbf{N}} \mathcal{M}_n$. A map $\tau : \Omega \mapsto \mathbf{N}\cup\{\infty\}$ is called a stopping time if

$$\{\tau = n\} \in \mathcal{M}_n \quad \text{for all } n \in \mathbf{N}. \tag{2.1}$$

Lemma 2.1. For a finite stopping time τ, we define

$$\tilde{X}_\tau(\omega) := \tilde{X}_n(\omega), \quad \omega \in \{\tau = n\} \quad \text{for } n \in \mathbf{N}. \tag{2.2}$$

Then, $\tilde{X}_\tau$ is a fuzzy random variable.

Let $g : \mathcal{I} \mapsto \mathbf{R}$ be a weighting function, which is continuous and monotone (see Fortemps and Roubens [3]). Using this g, the scalarization of the fuzzy reward will be done by

$$G_\tau(\omega) := \begin{cases} \displaystyle\int_0^1 g(\tilde{X}_{\tau,\alpha}(\omega))\,\mathrm{d}\alpha, & \text{if } \tau(\omega) < \infty \\ \displaystyle\limsup_{n\to\infty} \int_0^1 g(\tilde{X}_{n,\alpha}(\omega))\,\mathrm{d}\alpha & \text{if } \tau(\omega) = \infty. \end{cases} \tag{2.3}$$

Note that $g(\tilde{X}_{\tau,\alpha}(\omega)) \in \mathbf{R}$ and the map $\alpha \mapsto g(\tilde{X}_{\tau,\alpha}(\omega))$ is left-continuous on $(0,1]$, so that the right-hand integral of (2.3) is well-defined. From the linearity of the weighting function g, we define

$$E(G_\tau) := E\left(\int_0^1 g(\tilde{X}_{\tau,\alpha}(\cdot))\,\mathrm{d}\alpha\right) = \int_0^1 g(E(\tilde{X}_\tau)_\alpha)\,\mathrm{d}\alpha \quad \text{for stopping times } \tau. \tag{2.4}$$

Definition 2.1. A stopping time τ^* is called optimal if $E(G_{\tau^*}) \geq E(G_\tau)$ for all stopping times τ.

Define

$$Z_n(\omega) := \operatorname*{ess\ sup}_{\tau\,:\,\tau\geq n} E(G_\tau|\mathcal{M}_n) = \operatorname*{ess\ sup}_{\tau\,:\,\tau\geq n} E\left(\int_0^1 g(\tilde{X}_{\tau,\alpha}(\cdot))\,d\alpha|\mathcal{M}_n\right), \tag{2.5}$$

for $\omega \in \Omega,\ n \in \mathbf{N}$.

Lemma 2.2. *Define*

$$\sigma^*(\omega) := \inf\{n \mid G_n(\omega) = Z_n(\omega)\}, \quad \omega \in \Omega,$$

where the infimum of the empty set is understood to be $+\infty$. If $\sigma^* < \infty$, then σ^* is an optimal stopping time for Definition 2.1.

3. A fuzzy stopping problem

Definition 3.1. A fuzzy stopping time is a map $\tilde{\tau} : \mathbf{N} \times \Omega \mapsto [0,1]$ satisfying the following (i) – (iii):

(i) For each $n \in \mathbf{N}$, $\tilde{\tau}(n,\cdot)$ is $\mathcal{M}_n$-measurable.

(ii) For each $\omega \in \Omega$, $n \mapsto \tilde{\tau}(n,\omega)$ is non-increasing.

(iii) For each $\omega \in \Omega$, there exists an integer n_0 such that $\tilde{\tau}(n,\omega) = 0$ for all $n \geq n_0$.

In the grade of membership of stopping times, '0' and '1' represent 'stop' and 'continue' respectively. The following lemmas imply the properties of fuzzy stopping times.

Lemma 3.1.

(i) *Let $\tilde{\tau}$ be a fuzzy stopping time. Define a map $\tilde{\tau}_\alpha : \Omega \mapsto \mathbf{N}$ by*

$$\tilde{\tau}_\alpha(\omega) = \inf\{n \in \mathbf{N} \mid \tilde{\tau}(n,\omega) < \alpha\} \quad (\omega \in \Omega) \quad \textit{for } \alpha \in (0,1], \tag{3.1}$$

where the infimum of the empty set is understood to be $+\infty$. Then, we have:

(a) $\{\tilde{\tau}_\alpha \leq n\} \in \mathcal{M}_n \quad (n \in \mathbf{N})$;

(b) $\tilde{\tau}_\alpha(\omega) \leq \tilde{\tau}_{\alpha'}(\omega) \quad (\omega \in \Omega) \quad$ *if* $\alpha \geq \alpha'$;

(c) $\lim_{\alpha'\uparrow\alpha} \tilde{\tau}_{\alpha'}(\omega) = \tilde{\tau}_\alpha(\omega) \quad (\omega \in \Omega) \quad$ *if* $\alpha > 0$;

(d) $\tilde{\tau}_0(\omega) := \lim_{\alpha\downarrow 0} \tilde{\tau}_\alpha(\omega) < \infty \quad (\omega \in \Omega)$.

(ii) *Let* $\{\tilde{\tau}_\alpha\}_{\alpha\in[0,1]}$ *be maps* $\tilde{\tau}_\alpha : \Omega \mapsto \mathbf{N}$ *satisfying the above (a) – (d). Define a map* $\tilde{\tau} : \mathbf{N} \times \Omega \mapsto [0,1]$ *by*

$$\tilde{\tau}(n,\omega) := \sup_{\alpha\in[0,1]} \{\alpha \wedge 1_{\{\tilde{\tau}_\alpha > n\}}(\omega)\}, \quad n \in \mathbf{N},\ \omega \in \Omega. \tag{3.2}$$

Then $\tilde{\tau}$ *is a fuzzy stopping time.*

Let $g : \mathcal{I} \mapsto \mathbf{R}$ be a weighting function (see [3]). For a fuzzy stopping time $\tilde{\tau}(n,\omega)$, the scalarization of the fuzzy reward will be done by

$$G_{\tilde{\tau}}(\omega) := \int_0^1 g(\tilde{X}_{\tilde{\tau}_\alpha,\alpha}(\omega))\,\mathrm{d}\alpha, \quad \omega \in \Omega, \tag{3.3}$$

where $\tilde{\tau}_\alpha$ is defined by (3.1). Note that $g(\tilde{X}_{\tilde{\tau}_\alpha,\alpha}(\omega)) \in \mathbf{R}$ and the map $\alpha \mapsto g(\tilde{X}_{\tilde{\tau}_\alpha,\alpha}(\omega))$ is left-continuous on $(0,1]$, so that the integral of (3.3) is well-defined. From the linearity of the weighting function g, we define

$$E(G_{\tilde{\tau}}) := E\left(\int_0^1 g(\tilde{X}_{\tilde{\tau}_\alpha,\alpha}(\cdot))\,\mathrm{d}\alpha\right) = \int_0^1 g(E(\tilde{X}_{\tilde{\tau}_\alpha})_\alpha)\,\mathrm{d}\alpha \tag{3.4}$$

for fuzzy stopping times $\tilde{\tau}$.

Definition 3.2.

(i) Let $\alpha \in [0,1]$. A stopping time τ^* is called α-optimal if $g(E(\tilde{X}_{\tau^*})_\alpha) \geq g(E(\tilde{X}_\tau)_\alpha)$ for all stopping times τ.

(ii) A fuzzy stopping time $\tilde{\tau}^*$ is called optimal if $E(G_{\tilde{\tau}^*}) \geq E(G_{\tilde{\tau}})$ for all fuzzy stopping times $\tilde{\tau}$.

Define a sequence of subsets $\{\Lambda_n\}_{n=0}^\infty$ of Ω by

$$\Lambda_n := \{\omega \in \Omega \mid g(\tilde{X}_{n,\alpha})(\omega) \geq E(g(\tilde{X}_{n+1,\alpha})|\mathcal{M}_n)(\omega)\}, \quad n \in \mathbf{N}.$$

Assumption A (Monotone case).

$$\Lambda_0 \subset \Lambda_1 \subset \Lambda_2 \subset \Lambda_3 \subset \cdots \quad \text{and} \quad \bigcup_{n=0}^\infty \Lambda_n = \Omega.$$

In order to characterize α-optimal stopping times, let

$$\gamma_n^\alpha := \operatorname*{ess\,sup}_{\tilde{\tau}\,:\,\tilde{\tau}_\alpha \geq n} E(g(\tilde{X}_{\tilde{\tau}_\alpha,\alpha})|\mathcal{M}_n) \quad \text{for } n \in \mathbf{N}. \tag{3.5}$$

And we define a map $\tilde{\sigma}_\alpha^* : \Omega \mapsto \mathbf{N}$ by

$$\tilde{\sigma}_\alpha^*(\omega) := \inf\left\{n \mid g(\tilde{X}_{n,\alpha})(\omega) = \gamma_n^\alpha(\omega)\right\} \tag{3.6}$$

for $\omega \in \Omega$ and $\alpha \in [0,1]$, where the infimum of the empty set is understood to be $+\infty$. Then, the next lemma is given by Chow et al. [2].

Lemma 3.2 ([2, Theorems 4.1 and 4.5]). *Suppose Assumption A holds. Then, the following (i) and (ii) hold:*

(i) $\gamma_n^\alpha(\omega) = \max\{g(\tilde{X}_{n,\alpha})(\omega), \gamma_{n+1}^\alpha(\omega)\}$ a.a. $\omega \in \Omega$ *for* $n \in \mathbf{N}$.

(ii) *Let* $\alpha \in [0,1]$. *If* $\tilde{\sigma}_\alpha^* < \infty$ a.s., *then* $\tilde{\sigma}_\alpha^*$ *is* α*-optimal and* $E(\gamma_0^\alpha) = E(g(\tilde{X}_{\tilde{\sigma}_\alpha^*,\alpha}))$.

In order to construct an optimal fuzzy stopping time from α-optimal stopping times $\{\tilde{\sigma}_\alpha^*\}_{\alpha\in[0,1]}$, we need a regularity condition.

Assumption B (Regularity of fuzzy stopping times). A fuzzy stopping time $\tilde{\sigma}^*$ is called regular if the map $\alpha \mapsto \tilde{\sigma}_\alpha^*(\omega)$ is non-increasing for each $\omega \in \Omega$.

Under Assumption B, we can assume the left-continuity of the map $\alpha \mapsto \tilde{\sigma}_\alpha^*(\omega)$ and we can define a map $\tilde{\sigma}^* : \mathbf{N} \times \Omega \mapsto [0,1]$ by

$$\tilde{\sigma}^*(n,\omega) := \sup_{\alpha\in[0,1]} \min\{\alpha, 1_{\{\tilde{\sigma}_\alpha^* > n\}}(\omega)\}, \quad n \in \mathbf{N},\ \omega \in \Omega. \tag{3.7}$$

Theorem 3.1. *Suppose Assumptions A and B hold. Then* $\tilde{\sigma}^*$ *is an optimal fuzzy stopping time.*

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